Relevance of Chaos and Strange Attractors in the Samuelson-Hicks Oscillator

Jean-François Verne

Saint-Joseph University of Beirut, Faculty of Economics, Center of Documentation and Economic Research CEDREC, Lebanon

jean-francois.verne@usj.edu.lb

Abstract

In this paper, we look for the relevance of chaos in the well-known Hicks-Samuelson's oscillator model investigating the endogenous fluctuations of the national income between two limits: full employment income and under-employment income. We compute the Lyapunov exponent, via Monte-Carlo simulations, to detect chaos in the evolution of the income between the both limits. In case of positive Lyapunov exponent and large values of parameter (i.e. marginal propensity to consume and technical coefficient for capital), the evolution of income is seen to be chaotic. The model also may contain a quasi-periodic attractor that can be chaotic or not.

Keywords: Chaos; Attractor; Oscillator; Post-Keynesian; Business Fluctuations.

JEL Classification: C62; E12; E20; E32.

1. Introduction

The original linear model of accelerator-multiplier developed by P.A. Samuelson (1939) relies on a multiplier mechanism, which is based on a simple Keynesian consumption with a lag, and investment, depending on the variation in consumption (determined by the level of economic activity), which involves the accelerator mechanism. The combination of these two mechanisms gives rise to the Samuelson's oscillator.

In his paper, Samuelson explains how multiplier and acceleration generate business cycles and fluctuations in national income. To demonstrate his purpose, he chooses several values of the marginal propensity to consume and the marginal coefficient of capital. According to certain values of these parameters, the evolution of national income exhibits oscillations. These oscillations may be damped, perfectly regular or explosive. Although this model contains some valid elements regarding the explication of economic fluctuations, it is not able to produce lasting business cycles. Moreover, empirically observed values of its coefficients imply that the trajectory of income is unstable (Westerhoff, 2006).

Thus, improving the Samuelson' model, JR Hicks (1950) adds some changes by indicating that in a stationary state, induced as well as total net investment must be nil and gross investment must be equal to depreciation. Furthermore, he adds a floor (the under-employment income) and a ceiling (the full employment income) in this model and formulates a piecewise linear framework that can produce bounded oscillations. He also adds a geometric growth model that can be coupled with the business cycles. Some authors find that "quasi-periodic attractors" can occur in the basic Hicks model and other authors investigate the mathematical properties of such a model (Gallegati, Gardini, Puu and Sushko, 2003).

In this paper, we show that, even though nonlinearity is a necessary (but not sufficient) condition for the occurrence of chaos in dynamical systems, the Samuelson-Hicks model displays chaos for plausible and widely used parameters values. Thus, we search the relevance of chaos characterized by quasi-periodic attractors by using Monte-Carlo simulation to estimate the Average Lyapunov Exponent that is an indicator of the degree of chaos. This paper contains the following sections. Section 2 presents an overview of chaotic models and the Samuelson-Hicks oscillator as well. Section 3 shows the evolution of the income between the floor and the ceiling. Section 4 exhibits the relevance of chaos by estimating the Average Lyapunov Exponent with Monte-Carlo simulation. Section 5 analyzes the possibility of the quasi-periodic attractor occurrence and makes a comparison between chaotic evolution and periodic, damped or explosive oscillations of national income. Section 6 concludes.

2. Chaotic model in economics and Samuelson-Hicks oscillator: An overview

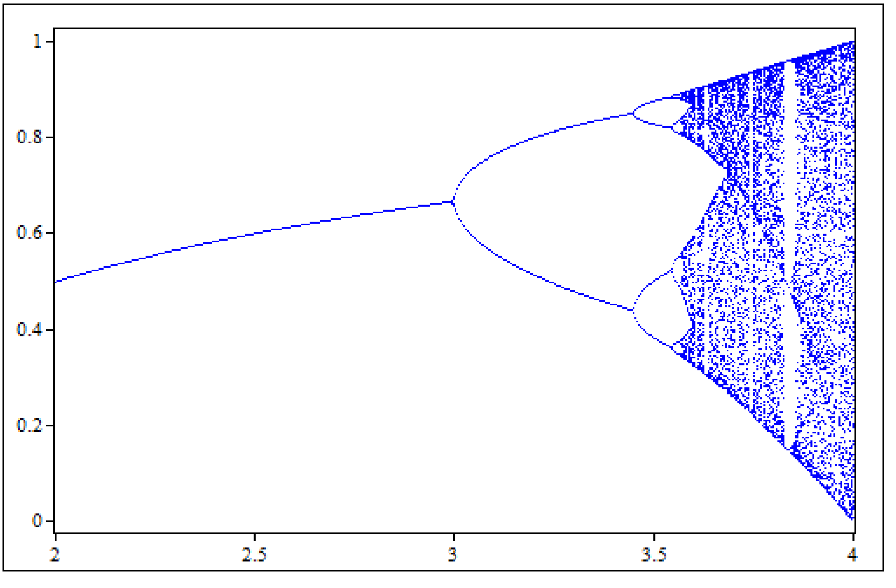
Chaos theory is primarily used in the meteorology fields (Lorenz, 1960, 1972). The main insight behind this concept is that even simple deterministic system can sometimes produce unpredictable situations notably when such deterministic system has a sensitivity to initials conditions in the short run.

Traditionally, chaos theory is analyzed by means of a logistic function used as a simple model of biological growth (Baumol et Benhabib, 1989) such as:

$y_{t+1} = ay_t(1 - y_t)$ with $0 < a < 4$ [1]

Figure 1 represents the evolution of the *x* variable (*y*-axis) as parameter *a* varies (*x*-axis).

Figure 1: Evolution of the system as parameter *a* varies



This Figure was shaped by simulating the evolution of the system over 10000 iterations. It shows that if a system exhibits repeated periods

doubling then it will have an infinite number of bifurcations in a finite increase of that parameter (Feigenbaum, 1978).
For $0 \leq a \leq 1$, the stationary solution at the origin is stable and the system exhibits a cycle of period 1. If $1 \leq a \leq 3$, the stationary solution is stable but when $a = 3.1$, the system undergoes a bifurcation and presents a cycle of period 2. If $a > 3.1$ (and equals around to 3.5) the cycle of period 2 splits into a cycle of period 4. From $a = 3.57$ to 4, the system adopts a chaotic behavior except between the values ranging from about 3.82 to about 3.86, where a white window appears. This indicates that the system moves from chaos back into order, but it bifurcates again and returns to chaos at $a = 3.86$.

The inclusion of this kind of model in economic analysis is not recent. For example, Mandelbrot (1963) analyses the chaotic variation of speculative prices. Kesley (1988), using the overlapping generation model, assert that economics models involve chaos. Baumol and Benhabib (1989) present the nonlinear models as an example of chaos estimation.
More recently, Viad et al., (2010), taking an example of chaos in exchange rates, show that chaos theory is related with the notion of nonlinearity. Federici and Gandolfo, (2014) propose a various tests of chaotic behavior in economics by also considering exchange rates. Other authors use chaos theory and the attractor approach to identify a chaotic dynamic in the evolution of GDP (Verne and Doueiry-Verne, 2019).

All these models are based on an econometric analysis taking account the random factor via residuals of equations. However, chaos theory can also be used in an endogenous fluctuations model that do not include the random factor.

Thus, in the Samuelson's original paper (1939, p. 76), we have four macroeconomic variables: the national income at time t; Y_t, which is the sum of three components: Governmental expenditure, G_t; consumption expenditure, C_t and private investment, I_t.

The first relationship between these four variables is an identity relation since we have, as in the Keynesian tradition:

$$Y_t, = C_t + I_t + A_t \quad [2]$$

With A_t exogenous (the autonomous expenditures).

In the Samuelson-Hicks model, investment in determined by the growth of income, through the principle of acceleration where investment is proportional to the rate of change in income:

$$I_t = k(Y_{t-1} - Y_{t-2}) \quad [3]$$

With k, the marginal coefficient of capital or the technical coefficient for capital e.g. the volume of capital needed to produce one unit of goods during one time period. Y_{t-1} and Y_{t-2}, are income of one and two periods back respectively.

The third relationship is about consumption expenditure function with the lagged income Y_{t-1}.

$$C_t = cY_{t-1} \quad [4]$$

With c, the marginal propensity to consume, we can write the national income as:

$$Y_t = (c + k)Y_{t-1} - kY_{t-2} + A_t \qquad [5]$$

From this relation, we can estimate the evolution of income according to the values of marginal propensity to consume and the technical coefficient for capital. For example, for large values of c and k, the national income records explosive oscillations while it presents perfectly periodic fluctuations when $k = 1$ and $c = 0.5$. If c and k parameters take certain values, we obtain the inverted complex roots from the relation [5] written in a polynomial form:

$$Y_t [1 - (c + k)L + kL^2] = A_t \qquad [6]$$

L is the lag operator where $L^k = Y_{t-k}$

Thus, in case of oscillations, the determinant is $\Delta = (c + k)^2 - 4k < 0$ and $L = \frac{(c+k)}{2} \pm \frac{i\sqrt{\Delta}}{2}$

Setting $\frac{(c+k)}{2} = \alpha$ and $\frac{i\sqrt{\Delta}}{2} = \beta$, we calculate the modulus $p = (\alpha^2 + \beta^2)^{0.5}$

If $p < 1$, the values of inverted roots are inside the unit circle of complex plane and income oscillations are damped. The process is stationary and the national income returns towards its long run value.

If $p > 1$, the values of inverted roots are outside the unit circle of complex plane and income oscillations are explosive.
If $p = 1$, national income oscillations exhibit perfectly sinusoidal fluctuations.

In the Hicks model, the lower limit (the floor) applied to induced investment while the upper limit (the ceiling) applied to full employment (Gallegati, Gardini, Puu and Sushko, 2003, p. 508). In addition, Hicks models a growth process by introducing autonomous expenditures, which may be growing exponentially i.e. $A_t = A_0(1 + g)^t$ where g is a given growth rate and A_0 a positive constant. Therefore, the solution of the characteristic equation with complex roots is the product of an exponential growth i.e. $Y_t = Y_0(1 + g)^t$.
By substituting the values of A_t and Y_t in [5] we define the stationary income and the two limits: The ceiling, e.g. the full employment income and the floor, the under-employment income where the induced investment is nil and gross investment equals to depreciation.
From relation [5] we can write:

$$Y_{\mathrm{t}} = (c + k)Y_0(1 + g)^{t-1} - kY_0(1 + g)^{t-2} + A_0(1 + g)^t \qquad [7]$$

By substituting $Y_t = = Y_0(1 + g)^t$ in [7], we have:

$$Y_0(1+g)^t - (c + k)Y_0(1 + g)^{t-1} + kY_0(1 + g)^{t-2} = A_0(1 + g)^t \qquad [8]$$

And:

$$Y_0(1+g)^{t-2}\,[\,(1+g)^2 - (c+k)(1+g) + k] = A_0(1+g)^t \qquad [9]$$

Finally, we obtain the stationary income or the equilibrium path:

$$Y_0 = \frac{A_0(1+g)^2}{[(1+g)^2-(c+k)(1+g)+k]} \qquad [10]$$

Relation [10] determines the equilibrium path around which the income Y_t may fluctuate.

In the Hicks model, we define the equilibrium growth path as:

$Y_E = Y_0(1+g)^t$

When the technical coefficient for capital $k > 1$, the national income leave the equilibrium path and inevitably reaches the ceiling of full employment for a maximum of two periods. Then, during the recession, national income falls on the floor.

The equation of the full employment path is: $YM_t = YM_0(1+g)^t$.

By substituting this term in the relation [5], we obtain:

$$Y_t = (c+k)YM_0(1+g)^{t-1} - kYM_0(1+g)^{t-2} + A_0(1+g)^t \qquad [11]$$

In fact:

$$(c+k)YM_0(1+g)^{t-1} - kYM_0(1+g)^{t-2} + A_0(1+g)^t < YM_0(1+g)^t \qquad [12]$$

Relation [12] is verified if:

$YM_0 > Y_0$ (computed in the relation [10]).

After two periods, a change in the trajectory of national income occurs. It is the beginning of the recession phase where the induced investment disappears due to the decline in production. Hence, $k = 0$ and the relation [5] is simplified:

$$Y_t = cY_{t-1} + G_t \qquad [13]$$

We define the under-employment path (the floor) as $YL_t = YL_0(1 + g)^t$. By using this term in the relation [13], we obtain:

$$YL_0(1 + g)^t = c\, YL_0(1 + g)^{t-1} + A_0(1 + g)^t$$

By rearranging the terms, we have:

$$YL_0(1 + g)^{t-1}(1 - g - c) = A_0(1 + g)^t \qquad [14]$$

Finally, we compute the under-employment income as follows:

$$YL_0 = \frac{A_0(1+g)}{1-g-c} \qquad [15]$$

During the recession phase, Y_t falls to the under-employment level.

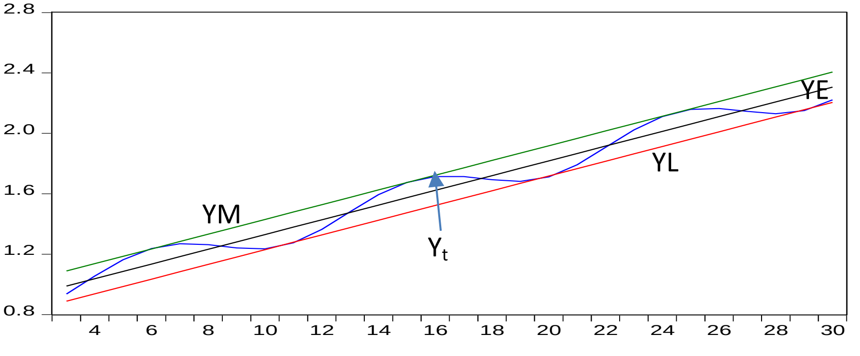
According to certain values of the marginal propensity to consume and the technical coefficient for capital, income displays several kinds of oscillations between the both limits.

3. Evolution of income between floor and ceiling

In order to display the evolution of income between ceiling and floor, we assume several values concerning the technical coefficient for capital, k and the marginal propensity to consume c. In addition, we take a period of 30 years and suppose that the economic growth rate g = 5% per year.

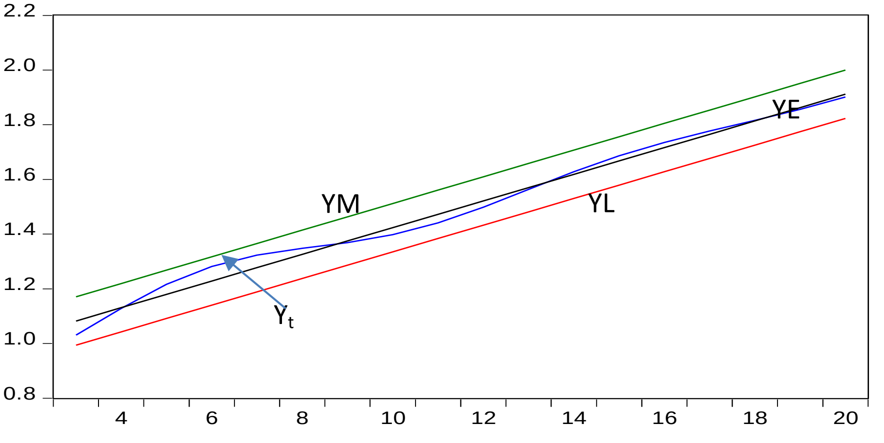
If we take the special case where c = 0.5 and k = 1, the evolution of income is seen to be perfectly sinusoidal between the both limits.

Figure 2: Sinusoidal evolution of income between ceiling and floor



In the Hicks model, the economy is not stationary and exhibits a positive growth rate. As long as $c < 0.6$ and $k < 1$, the fluctuations of the national income Y_t remain inside the both limits and are damped as the Figure 3 displays it.

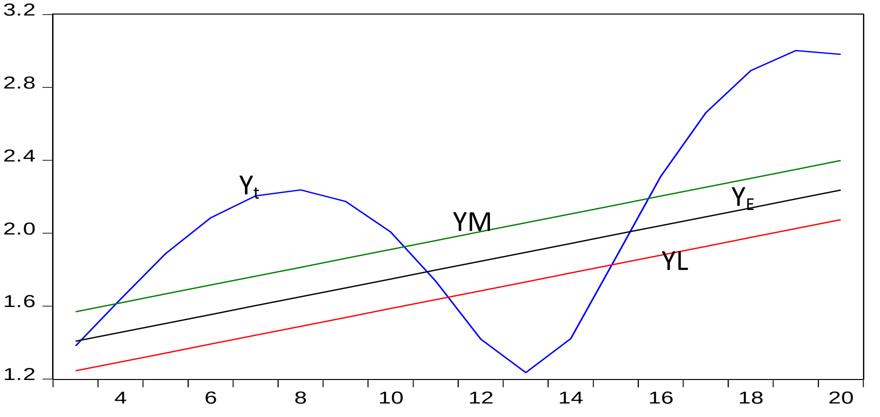
Figure 3: Damped fluctuations



For example, if $c = 0.6$ and $k = 0.8$, the fluctuations of income are damped (the national income is running towards its equilibrium value) and remain inside the corridor as long as the marginal propensity to consume is less than 0.6.

However, when $c > 0.6$ and $k > 1$, oscillations in national income become explosive.

Figure 4: Explosive fluctuations



For c = 0.7 and k = 1.2, explosive fluctuations of income are out of the upper and lower limits.

The Samuelson-Hicks model can exhibit chaos because it implies a second-order difference equation for output. This arises because investment is assumed to depend on the lagged change in output. The key mechanism highlighted by Samuelson is the accelerator effect, which arises because investment depends on the change in output. The assumption that investment depends on the lagged change in output is not essential; the accelerator effect also arises if investment depends on the current change in output. But in that case, chaos does not arise as output is a first-order difference equation, not second-order. Thus, if output is a second-order equation, the occurrence and relevance of chaos, measured by the Lyapunov exponent, depend on the values of capital coefficient (k) and marginal propensity to consume (c)

4. The Lyapunov exponent

The Lyapunov exponent is the quantity that characterizes the rate of separation of infinitesimal close trajectories. It plays an important role in identifying of the chaotic degree of the strange attractor (Wu and Baleanu, 2015). The number of Lyapunov exponents equals the number of state variables considered. If we consider a unidimensional system, we may compute one single exponent (Lopez-Jéminez et al., 2002).

A positive Lyapunov exponent does cause this separation to increase over further iterations and shows a chaotic dynamic. A negative Lyapunov exponent indicates an attracting fixed point or periodic cycle and implies non-chaotic dynamic characterized by a strange non-chaotic attractor. A Lyapunov exponent equals to zero displays sinusoidal oscillations and periodic attractor.

For example, for searching chaos in the Hicks model, we use the Wolf method (1985) to estimate the Lyapunov exponent (called λ_t) with different values of the marginal propensity to consume *c* and the marginal coefficient of capital *k*.

By this method, we start from an initial condition Y_t in the Hicks model and we consider a very close value of separation, where the initial distance d_0 is extremely small. The absolute value of d_t after *t* iteration is:

$$|d_t| = |d_0|e^{\lambda t} \qquad [16]$$

It is equivalent to write:

$$\lambda_t = \lim_{t\to\infty} \frac{1}{t}\left|\frac{\partial d(t)}{\partial do}\right| \qquad [17]$$

We choose the value of the separation $d_0 = 10^{-4}$ and obtain values of λ_t that give the values of the Lyapunov Exponent. After a Monte Carlo simulation with 1000 random values of coefficient for capital, k (ranging from 0 to 4), we estimate the Average Lyapunov Exxponent (*ALE*). Since chaos arises, as output is a second order difference equation, the marginal propensity to consume (include in the first-order difference equation) is fixed. It takes several values (0.5, 0.6 and 0.8) in the Samuelson's original paper (1939, p. 77). We arbitrarily choose $c = 0.8$.

Figure 5: *ALE* evolution with respect to coefficient for capital

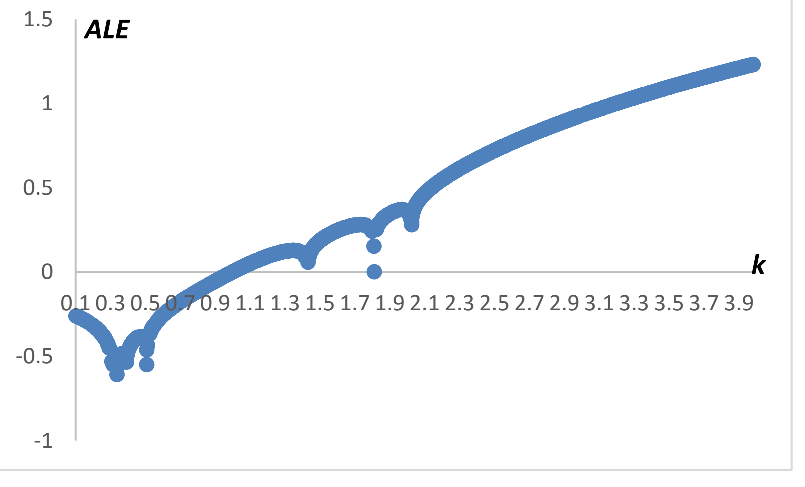


Figure 5 shows that the *ALE* is negative for $0 < k < 1$. This means that the behavior of the national income (Y_t) exhibits non-chaotic dynamic characterized by damped oscillations (stationary process with modulus ρ < 1). Then, for k = 1, the *ALE* is nil meaning that the national income fluctuations are sinusoidal (with the modulus ρ = 1). It becomes more relevant when k > 1. This means that values in the region k > 1 are much more likely to lead to chaos. However, as Feigenbaum (1978) shows it in Figure 1 (where a parameter ranges from about 3.82 to about 3.86), for k = 1.5, *ALE* = 0. This means that national income moves from chaos back into order and returns to chaos at k > 1.5. From k > 1.5 to k = 4, the national income exhibits an increasingly chaotic dynamic. In such a region, the oscillations are explosive and the Lyapunov exponent is strongly positive (with modulus ρ >1).

According to the values of the technical coefficient for capital, which is the key parameter leading the national income to chaos, we can observe the occurrence of several attractors inside or outside the both limits.

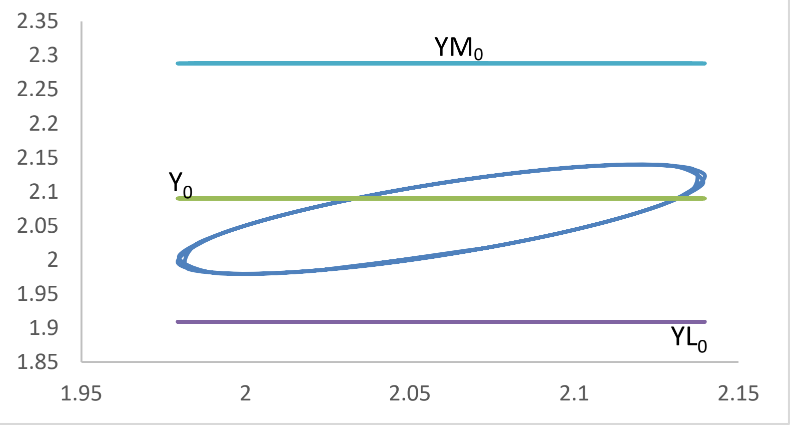
5. Quasi-periodic attractors in the Hicks model

Chaos theory involves the concept of the strange attractor for which the trajectories of a variable have a bizarre structure, being nether simple smooth, nor continuous curves but fractals (Puu, 1997). Fractals (Mandelbrot, 1982) could be an indefinite set of unconnected points or a smooth curve with mathematical discontinuity or curve that is fully connected but discontinuous everywhere.

In fact, we have quasi-periodic attractor when every trajectory winds around endlessly on a torus (Strogatz, 1994).

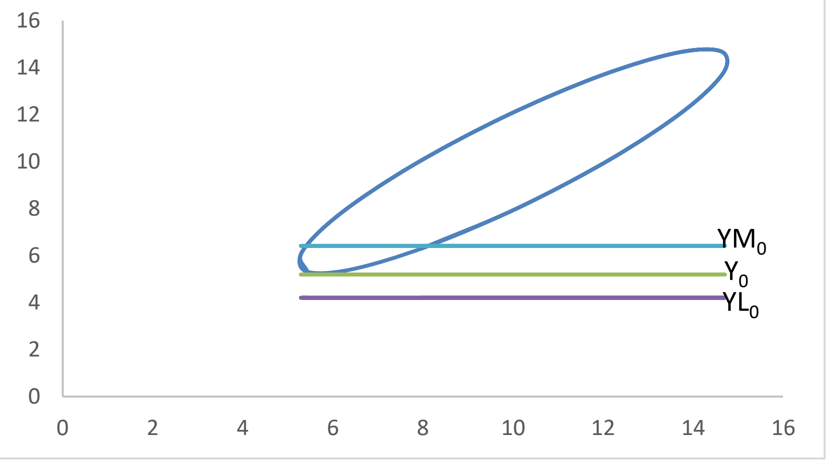
Thus, the following Figures represent the strange attractor showing the national income evolution in the space phase where each ordered pair (Y_{t},, $Y_{t\text{-}1}$; t = 2, …, N) is displayed in the plane (Figures 6). The y-axis represents the values of Y_t and x-axis, values of $Y_{t\text{-}1}$ (Kriz, 2011). The three levels of income (equilibrium income, full employment income and under-employment income) are represented as well.

Figure 6-a: National income in the space phase: The perfectly periodic attractor between the both limits



This Figure shows a perfectly periodic attractor between the upper limit (the income of full employment, called YM_0) and the lower limit (the income of under-employment, called YL_0). Thus, when c = 0.5 and k = 1, the modulus $\rho = 1$ (e.g. that oscillations are perfectly sinusoidal) and the periodic attractor is inside the both limits. In addition, the Lyapunov exponent is nil meaning that a periodic attractor occurs. However, a rise in the propensity to consume (the coefficient of capital remaining equal to one), pushes the periodic attractor out of the upper limit (Figure 6-b).

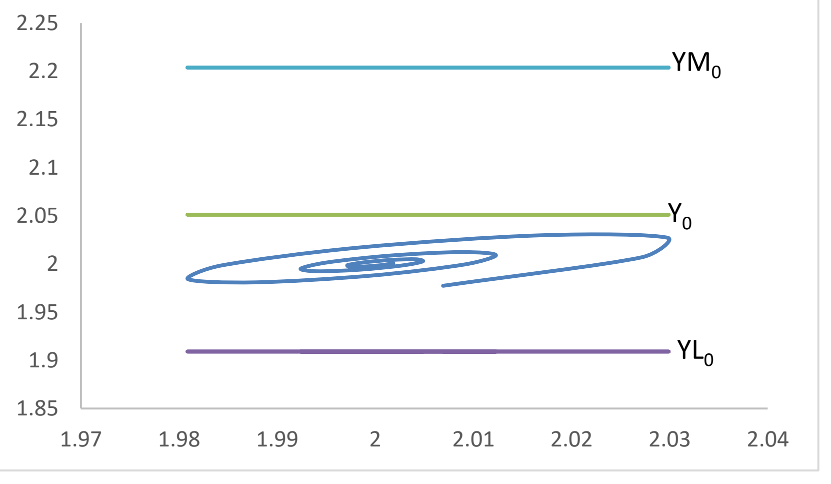
Figure 6-b: National income in the space phase: The perfectly periodic attractor out of the upper limit



This Figure exhibits the case where $c = 0.8$ and $k = 1$.

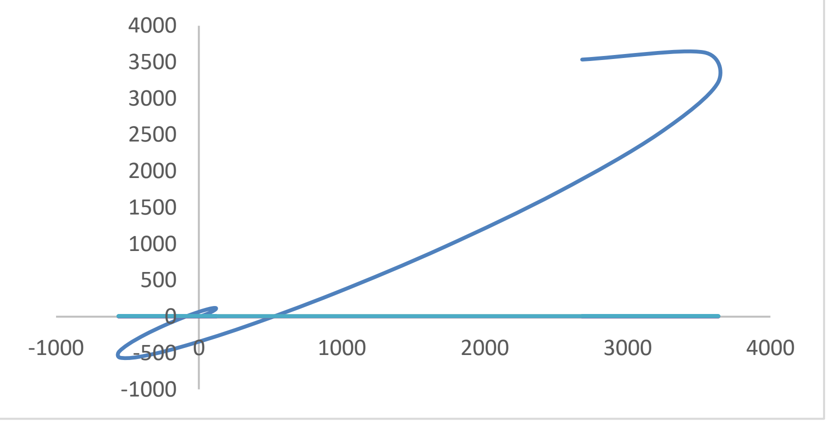
As long as $c \leq 0.5$ and $k = 1$, we have a perfectly sinusoidal oscillations and periodic attractor inside the both limits. But, if the technical coefficient for capital become less than one (with $c \leq 0.5$), the fluctuations are damped (the modulus $p < 1$), and the figure exhibits a strange non-chaotic attractor taking the form of an ellipsoid (Figure 6-c).

Figure 6-c: National income in the space phase: The occurrence of a strange non chaotic attractor



This Figure shows that even though the national income exhibits explosive fluctuations in the short run, a strange non-chaotic attractor does exist in the long run that pushes income to regain regular growth. In other words, the national income enters the ellipsoid and then remains trapped therein for all future time (Hirsh, Smale and Devaney, 2004). However when $c > 0.5$ and $k > 1$, the national income records explosive fluctuations and moves away his trajectory (Figure 6-d).

Figure 6-d: National income in the space phase: The go out of the trajectory



This Figure shows the case where c = 0.8 and k = 1.6 and displays a chaotic strange attractor that goes beyond both limits. In this hypothesis, the national income that starting far from the origin goes away the ellipsoid and does not return on the equilibrium path. The trajectory of income moves away from the ellipsoid for all future time.

In addition, all figures exhibit a periodic or quasi-periodic attractor (that can be chaotic or not) when the national income records oscillations e.g. when the determinant of the polynomial equation Δ is negative. On the contrary, if $\Delta > 0$ (when the parameters c = 0.8 and $k > 3$), the evolution of national income becomes explosive without oscillations and the quasi-periodic attractor disappears.

6. Conclusion

In the Samuelson-Hicks model, the oscillations of income between two limits, e.g. the full employment income and under-employment income, depend on the values of the marginal propensity to consume and the technical coefficient for capital. However, the coefficient for capital is the key parameter explaining the relevance of chaos as output is a second order difference equation.

Furthermore, according to some values of the Average Lyapunov Exponent (*ALE*), a strange attractor exists and may be chaotic or not.

When the *ALE* is negative, the system has an attracting fixed point or periodic cycles characterized by a strange non chaotic attractor localized between the both limits. When the *ALE* is null, the system displays perfectly sinusoidal fluctuations inside the both limits and presents a perfectly periodic attractor. Chaos and explosive oscillations may occur with certain high values of the two parameters for which the determinant of the polynomial equation remains negative. In such a hypothesis, the *ALE* becomes positive and the income moves out of equilibrium. Moreover, the attractor becomes chaotic and moves outside both limits. This means that in the Hicks-Samuelson model, the relevance of chaos depends on values taken by the coefficient for capital. For lower values, income oscillations are damped and the attractor is between the two limits. In addition, the *ALE* is negative and the strange attractor pushes income to regain regular growth. This illustrates a strange non-chaotic attractor where the income enters the ellipsoid.

The attractor and the oscillations disappears when the determinant of the polynomial equation is positive e.g., when the marginal propensity to consume and the coefficient of capital reach larger values than in the aforementioned case.

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