Diagnosis of Financial Crisis by High Moment Deviations and Changing Transition Probability

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Abstract

High moments and transition probability provide new tools for diagnosing a financial crisis. Both calm and turbulent markets can be explained by the birth-death process for up-down price movements driven by identical agents.

The master equation approach in statistical mechanics and social dynamics derives the time-varying probability distribution for population dynamics. We calculate the transition probability from stock market indexes from open economic systems that is different from a Gaussian distribution from equilibrium physics in conservative systems. Market instability can be observed from dramatically increasing 3rd to 5th moments before and during the crisis and the changing shape of transition probability. Positive and negative feedback trading behavior can be revealed by the upper and lower curves in transition probability. There is a clear link between liberalization policy and market nonlinearity. The crisis condition for market breakdown can be found from the solution of the nonlinear birth-death process. Numerical estimation of a market turning point is close to the historical event of the U.S. 2008 financial crisis. The representative agent model of random walk and geometric Brownian motion in macro and finance theory does not provide any clue on the condition of a financial crisis. The sub-prime crisis in the U.S. indicates the limitation of a diversification strategy based on a mean-variance analysis. Historical lessons in recurrent financial crises demand a more general framework with endogenous instability and economic complexity, such as nonlinear interactions, non-stationary dynamics, and social interaction. Our general framework
greatly extends the scope of equilibrium models in finance, which is a special case of a calm market in our nonlinear model. We obtain a unified picture of economic complexity with endogenous instability and market resilience (JEL G01, E32, C58)

**Key Words:** financial crisis, high moments, transition probability, master equation, birth-death process, crisis condition

How to diagnose the nature of business cycles and financial crisis is still an open issue in economics and finance (Reinhart and Rogoff, 2009; Chen 2010). There are two schools of thoughts in studying business cycles and crisis. The equilibrium-exogenous school attributes external shocks as the only source of cycles and crisis (Frisch, 1933; Friedman and Schwartz, 1963; Lucas, 1981; Kydland and Prescott, 1982; Bernanke, Gertler and Gilchrist, 1996; Barro and Ursúa, 2008), while the disequilibrium-endogenous school mainly considers the internal mechanism of market instability (Keynes, 1936; Samuelson, 1939; Goodwin, 1951; Hayek, 1975; Diamong and Dybvig, 1983; Minsky, 1986). The two schools have fundamental differences in policy implications. There is little room for government interference in the exogenous theory; but government may play a key role in a macro economy for the endogenous school.

In modeling strategy, three approaches are popular in studying financial crisis. The first is the multi-equilibrium model with finite periods (Diamong and Dybvig, 1983; De Long, Shleifer, Summers, and Waldmann, 1990). This type of model is simple enough to demonstrate some stylized facts of market instability, but limited in use for empirical analysis because of the rigid time-length of different market regimes. The second approach is the representative agent model under external shocks. Its popularity in econometrics is based on its mathematical simplicity in statistical assumptions. Conflicting pictures result from different formulations of static distribution. The equilibrium school assumes the Gaussian distribution or i.i.d. with finite mean and variance (Fama, 1970), while the disequilibrium school introduces
non-Gaussian distributions, such as Levy distribution, fat tails and power law (Mandelbrot, 1963; Laherrère and Sornette, 1999; Mantegna and Stanley, 2000; Barror and Ursúa, 2008; Gabaix, Gopikrishnan, Plerou and Stanley, 2006). This approach is attractive in empirical analysis but impotent in policy studies. The third approach is a computational simulation of heterogeneous agents (Hommes, 2006). This approach can generate unstable patterns suggested by behavioral economics, but are hard to apply in empirical analysis, since its mathematical structure is quite complex. The issue in scientific strategy is to choose a proper degree of abstraction, so that it is simple enough to explain key empirical observations but complex enough to derive clear theoretical implications.

From a methodological perspective, the controversy between exogenous and endogenous schools can be solved by empirical investigation if the analytical framework is general enough to include both cases as its special cases. In economic analysis, equilibrium images are mainly constructed through four building blocks. First, the concept of an equilibrium state means that history does not matter in economic affairs. This situation can be described by a time series with only short correlations, such as the random walk or Brownian motion models. This state can be achieved by the first differencing in econometrics. Second, an equilibrium state implies a stationary time series, which can be approximated by the log-linear growth trend in macro or risk-free interest rate in finance. Third, the arbitrage-free portfolio can be constructed in finance theory if the market is dominated by linear pricing (Ross, 1976). Fourth, the complex nature of social interaction can be ignored if a many-body problem (such as more than three interacting agents and population with many agents) can be treated as the one-body problem, such as the case of the representative agent model. Therefore, we can develop a more general framework so that the equilibrium, stationary, and linear scenario can be extended to non-equilibrium, non-stationary, and nonlinear situations. The key step is replacing the static model of the representative agent by the time-varying probability distribution of population dynamics. This approach is widely used in statistical mechanics in physics and chemistry, which is called the master equation approach or
social dynamics in studies of interacting agents in economics (Weidlich and Braun, 1992; Lux, 1995; Faller and Petruccione, 2003; Aoki, 2004). According to this approach, equilibrium analysis mainly considers the first (mean value) and second moment (variance), while non-equilibrium situations also study social behavior with higher moments (Alfarano, Lux and Wagner, 2008). We will see that high moments in probability distribution are closely associated with a nonlinear mechanism in economic dynamics, which is the origin of endogenous instability and crisis. We will demonstrate that the time-varying probability distribution is richer than the static feature from the representative agents, but simpler than the computational model of heterogeneous agents in empirical and theoretical analysis.

Recent events of the financial crisis shed new light to address the above fundamental issues. The equilibrium models of efficient market and option pricing are based on the representative agent model of random walk and geometric Brownian motion. They do not provide any clue on the condition of a financial crisis. The 2008 financial crisis originated from the sub-prime crisis in the U.S. which indicates the limitation of a diversification strategy based on the mean-variance analysis. Historical lessons in recurrent financial crises demand a more general framework with endogenous instability and economic complexity, such as nonlinear interactions, non-stationary dynamics, and collective behavior. We introduce the master equation approach and the birth-death process as the general model of financial dynamics, which describes the up-down price movements driven by trading among a population of identical agents. The application of statistical mechanics in physics to social dynamics models in economics and finance introduces the time-varying probability distribution for population dynamics. The problem is how to derive the transition probability from empirical observation and economic mechanism since the Gaussian type distribution in equilibrium physics may not be valid for social systems (Prigogine, 1980; Chen, 1991). We need a new understanding of the seemingly contradictory phenomena of market resilience and recurrent crisis.

Our work started from an empirical time series analysis of stock market indexes. In this paper, we do not make any ad hoc behavioral assumptions for market dynamics.
Instead, we take a phenomenological approach to derive the transition probability from empirical data. We develop a numerical algorithm to bridge the gap between the master equation and the empirical estimation of transition probability. Most authors assume the transition probability follows the Gaussian distribution, which originated from equilibrium statistical mechanics (Weidlich and Braun, 1992). We pointed out that social interaction may lead to a logistic function in transition probability (Chen, 1991, 2010). We estimate the transition probability in two separate periods: one period of 1950 to 1980, was dominated by Keynesian policy and New Deal regulation; and the period of 1981 to 2010 is the liberalization era started by the Reagan-administration and includes the 2008 crisis. In each period, we assume that the transition probability is the function of market states (i.e. current prices), but independent of time. This procedure is similar to the two-stage econometric analysis. The difference is attributed to different mathematical representations. Econometric analysis is based on a matrix, while a probability distribution needs to solve partial differential equations. A more advanced mathematical representation may reveal more patterns in complex dynamics.

Through empirical analysis, we introduced three quantitative indicators of population behavior: (a) the relative deviation of a positive time series for macro and financial indexes with nonlinear trend; (b) the high (2\textsuperscript{nd} to 5\textsuperscript{th}) moment of time series measured by a moving time window; and (c) the transition probability between neighboring states of price indexes. We discovered three features from the US economic time series that reveal the complex features of business cycles and economic crisis: (A) the stable pattern of the relative deviation that cannot be explained by the representative agent model, such as random walk or Brownian motion (Chen, 2002, 2005; Li 2002); (B) the dramatic rise (1000 times or more) of high (3\textsuperscript{rd} to 5\textsuperscript{th}) moments before and during the crisis that signals the breakdown of the financial market and a failure of portfolio diversification strategy (Chen, 2010); and (C) the nonlinear shape of transition probability for the period of liberalization and crisis, which is rooted in trading behavior. We found a visible link between the liberalization policy and the financial crisis. This result is very different from the
exogenous school (Laherrère and Sornette, 1999; Barror and Ursúa, 2008; Gabaix, Gopikrishnan, Plerou and Stanley, 2006).

Through theoretical modeling, we demonstrate that the birth-death process is the proper model in population dynamics, which is capable of explaining all three observed features. The birth-death process originated in molecular dynamics in physics and has been introduced to describe an up-down process in stock price movement (Reichl, 1998; Cox and Ross, 1976; Kou and Kou, 2002; Zeng and Chen, 2008). We use the birth-death process as a unified model of calm and turbulent markets. By means of moment expansion, we estimated the condition of market breakdown, which is remarkably close to the real event. Unlike the model of heterogeneous agents, our population model of identical agents provides an alternative picture of animal spirits. Mass psychology is visualized by the rising and falling market tide that is measured by the net daily change rate.

Based on these findings, we get a new understanding of old conflicting thoughts. The so-called efficient market provides a simplifying picture of the calm market, whose higher moments are much smaller than the variance. The turbulent market during crisis resulted from the rise of high moments when the buy and sell pattern is remarkably nonlinear and asymmetric. There is strong evidence of endogenous instability, since the financial market is resilient under repeated cycles and crisis, which is characterized by the stable regime of the relative deviations (Chen, 2002, 2005, 2010). Our picture greatly extends the scope of equilibrium models in finance, which can be considered as a special case of a calm market in our nonlinear model.

The paper is organized as follows. Section I gives the data sources and main empirical observations, including the stable pattern of relative deviation and changing behavior of high moment deviations for calm and turbulent periods. Section II introduces the theoretical models, including the master equation and the birth-death process in continuous-time and discrete state space. Section III estimates transition probability from empirical data. Section IV investigates the critical condition of market break-down. Section V presents the main conclusions.
I. Data and Empirical Observations

The data we chose are standard in analyzing the U.S. business cycles and economic crisis. Our main purpose is to develop new mathematical representation for empirical and theoretical analysis. Three market indexes are used to demonstrate our approach: (1) the Dow-Jones Industrial average (DJI) index published by Wharton Research Data Services (WRDS 1915-2010) and www.yahoo.com (1930-2010); (2) S&P 500 index published by www.yahoo.com (1950-2010); and (3) the interest rate spread between three-month Eurodollar LIBOR rate and 3 month U.S. Treasury Bill rate (TED spread) published by Bloomberg (1990-2010). The sampling frequency is selected according to the needs of the theoretical framework and empirical algorithm. For example, monthly data is used for analyzing the trend and fluctuations in the frequency range of NBER business cycles. The daily data is used for estimating high moments and transition probability.

A. Trend, Cycles, and Relative Deviation

One essential feature of macro and financial indexes is their growing trend in addition to erratic cycles and fluctuations. The conventional measurement of business fluctuations in econometrics is using the first differencing. Its implicit assumption is ignoring the impact of a trend in time series analysis. The HP filter is introduced to specify the medium-term trend and cycles around the trend (Hodrick and Prescott, 1997).

Numerically, the trend series can be defined by the HP filter, which satisfies the following algorithm:

\[
\min_{\{x\}} \left\{ \sum_{t=1}^{T} (x_t - E(x_t))^2 + \lambda \sum_{t=1}^{T} [(E(x_{t+1}) - E(x_t)) - (E(x_t) - E(x_{t-1}))]^2 \right\},
\]
Here, \( x = Y(t) \) is a real economic variable (such as the stock price in option pricing model). For logarithmic data \( x = S = \ln Y \) when time series has a visible growth trend. For NBER business cycles, \( \lambda \) is 400 for annual data, 1600 for quarterly data, and 14400 for monthly data.

If we denote the original time series as \( S(t) \), the logarithmic time series \( X(t) \), its trend series \( G(t) \), and its cycle series \( C(t) \), we have:

\[
X(t) = \log S(t)
\]

\[
X(t) = G(t) + C(t)
\]

We introduced a useful measurement of relative deviation for a positive macro and financial index, which characterizes the relation between the standard deviation of the cyclic series and the mean of trend series.

[Definition 1]

We define the relative deviation (RD) as the ratio of the standard deviation of the cycle series \( C(t) \) to the mean of the trend series \( G(t) \) observed within a moving time window when \( G(t) \) is a positive series:

\[
RD = \Omega_N = \frac{\text{std}[X_N]}{\text{mean}[X_N]}
\]  

In statistics, this ratio is called the variation coefficient without restriction to the range of mean series. We call it the relative deviation only for positive variables, since the standard deviation can be considered as the absolute deviation. This measurement is very useful for characterizing the micro foundations in theoretical biology and persistent fluctuations in sustainable markets (Schrödinger, 1944; Chen, 2002).

The empirical pattern of a macro or financial index is demonstrated in Figure 1 to 3.
Figure 1. The original DJI (Dow Jones Industrial average) index monthly series (Oct. 1928 – June 2011). The peak of the 2000 internet bubble and the dip of the 2008 financial crisis are visible from the plot.

Figure 2. The logarithmic series of DJI (dotted series) and its smoothed HP growth trend (solid line) defined by the HP filter.
Figure 3. The relative deviations of DJI monthly series observed from a moving time window of 5 years.

In Figure 3, we can see that the relative deviation of DJI is reduced to more than half of its 1930s magnitude after World War II. A stable pattern of RD is visible without damping or explosive trend after 1950, which is the empirical foundation for our theoretical investigation in Section II.

**B High Moment Deviations before and during the Financial Crisis**

[Definition 2]

We define the $k$ th un-annualized moment as $\frac{\sum_{i=1}^{N} (x_i - E(x))^k}{N}$. 
In the following tables, we denote $\sigma^2$ as variance, $\mu^3$ the 3rd moment, $\nu^4$ the 4th moments and $\xi^5$ the 5th moment in later calculations.

In numerical estimation of $E(x)$, we take

$$\frac{\sum_{i=1}^{N} (x_i - E(x_i))^k}{N} = \frac{1}{2} \left( \frac{\sum_{i=2}^{N} (x_i - x_{i-1} - \mu_{i-1})^k}{N - 1} \right),$$

We use $\frac{\sum_{i=2}^{N} (x_i - x_{i-1})^k}{N - 1}$ to calculate the $k$ th un-annualized moment because the daily trend $\mu$ is small. In calculating the quarterly moments of the Dow-Jones industry daily index and TED spread, each quarter contains approximately 61 trading days, so that $N = 61$. The moments from 2nd to 5th are given in Table 1. Their dynamic patterns are shown in Figure 4 and Figure 5.
Table 1. The moments (unannualized) of logarithmic daily close of Dow-Jones industry index in the crisis

<table>
<thead>
<tr>
<th></th>
<th>4&lt;sup&gt;th&lt;/sup&gt;/1973</th>
<th>4&lt;sup&gt;th&lt;/sup&gt;/1987</th>
<th>4&lt;sup&gt;th&lt;/sup&gt;/1929</th>
<th>4&lt;sup&gt;th&lt;/sup&gt;/2008</th>
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<tr>
<td></td>
<td>normal value*</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt;</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt;</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt;</td>
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<tr>
<td>variance</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt; = 1 × 10&lt;sup&gt;-4&lt;/sup&gt;</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt; = 1 × 10&lt;sup&gt;-4&lt;/sup&gt;</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt; = 1 × 10&lt;sup&gt;-4&lt;/sup&gt;</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt; = 1 × 10&lt;sup&gt;-4&lt;/sup&gt;</td>
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<td>in previous quarter</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt; ~ σ&lt;sup&gt;2&lt;/sup&gt;</td>
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<td>σ&lt;sup&gt;2&lt;/sup&gt; ~ σ&lt;sup&gt;2&lt;/sup&gt;</td>
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<td>in crisis quarter</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt; ~ 2σ&lt;sup&gt;2&lt;/sup&gt;</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt; ~ 18σ&lt;sup&gt;2&lt;/sup&gt;</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt; ~ 19σ&lt;sup&gt;2&lt;/sup&gt;</td>
<td>σ&lt;sup&gt;2&lt;/sup&gt; ~ 17σ&lt;sup&gt;2&lt;/sup&gt;</td>
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<td>3rd moment σ&lt;sup&gt;3&lt;/sup&gt;</td>
<td>normal value</td>
<td>σ&lt;sup&gt;3&lt;/sup&gt; ~ -10&lt;sup&gt;-3&lt;/sup&gt;σ&lt;sup&gt;2&lt;/sup&gt;</td>
<td>σ&lt;sup&gt;3&lt;/sup&gt; ~ -10&lt;sup&gt;-3&lt;/sup&gt;σ&lt;sup&gt;2&lt;/sup&gt;</td>
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<td>σ&lt;sup&gt;4&lt;/sup&gt; ~ 10&lt;sup&gt;-4&lt;/sup&gt;σ&lt;sup&gt;2&lt;/sup&gt;</td>
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<td>in crisis quarter</td>
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<td>σ&lt;sup&gt;4&lt;/sup&gt; ~ 2 × 10&lt;sup&gt;6&lt;/sup&gt;σ&lt;sup&gt;2&lt;/sup&gt;</td>
<td>σ&lt;sup&gt;4&lt;/sup&gt; ~ 9 × 10&lt;sup&gt;5&lt;/sup&gt;σ&lt;sup&gt;2&lt;/sup&gt;</td>
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<td>5th moment σ&lt;sup&gt;5&lt;/sup&gt;</td>
<td>normal value</td>
<td>σ&lt;sup&gt;5&lt;/sup&gt; ~ 10&lt;sup&gt;-4&lt;/sup&gt;σ&lt;sup&gt;2&lt;/sup&gt;</td>
<td>σ&lt;sup&gt;5&lt;/sup&gt; ~ 10&lt;sup&gt;-4&lt;/sup&gt;σ&lt;sup&gt;2&lt;/sup&gt;</td>
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<tr>
<td>duration (trading days)</td>
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<td>121</td>
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</table>

*The normal magnitude of variance is about 10<sup>-5</sup> to 10<sup>-4</sup>. Here we chose σ<sup>2</sup> = 10<sup>-4</sup> as the normal level. Therefore, we would consider high moments when they reach the level of 10<sup>-2</sup>σ<sup>2</sup> or higher.
(b)

(c)
Figure 4. The quarterly moments of the of Dow-Jones Industrial Average (DJI) index.

Note: $S(t)$ is the natural logarithmic daily close price series. Each quarterly moment is computed with 61 daily data within 3 months. The data source is: www.yahoo.com, WRDS. Plots (a), (b), (c) and (d) are corresponding to $2^{nd}$, $3^{rd}$, $4^{th}$ and $5^{th}$ moment, respectively. The dashed line is the log(dji) series. The solid line is the trajectory of the n-th moment where n=2, 3, 4, 5.

In Figure 4, we can see that the sharp peaks appear during turbulent periods in the financial market. Four dramatic peaks can be identified in their corresponding periods: the quarters of $4^{th}$/1929, $4^{th}$/1931-4$^{th}$/1933, $4^{th}$/1987, and $4^{th}$/2008. All major peaks occurred during the historical crisis. Certainly, there are small peaks in addition to major market fluctuations.

Five patterns can be observed from Figure 4 and Table 1.

First, the magnitude of the high moments ($3^{rd}$ to $5^{th}$ moment) are quite small (say, less than 0.1% to 0.001% of the magnitude of the variance) compared to the $2^{nd}$ moment during the periods of a calm market. This observation shows that the mean-variance model in neoclassical finance theory is a good approximation only for
the calm market when the higher than 2\textsuperscript{nd} moments can be ignored (Markowitz, 1952).

Second, in the quarters before and in the crisis period, the magnitudes of the higher moments typically increase 100 to 1000 times, so that the magnitudes of high moments are comparable to or even larger than the usual magnitude of variance. This observation is true for 3\textsuperscript{rd}/1914, 4\textsuperscript{th}/1929, 4\textsuperscript{th}/1931-4\textsuperscript{th}/1933, 4\textsuperscript{th}/1987, 4\textsuperscript{th}/2008. There are less sharp peaks at 1\textsuperscript{st}/1907, 3\textsuperscript{rd}/1939, 2\textsuperscript{nd}/1940 and some small peaks around 2000. Therefore, mean-variance analysis and the Black-Scholes option pricing model (Black and Scholes, 1973) would breakdown before and during market turmoil, since the variance is not constant and high moments cannot be ignored in financial analysis. That is why a portfolio diversification strategy and a derivative market could fail during the crisis. This observation is beyond the scope of linear models of financial theory.

Third, the magnitudes of higher moments alone cannot tell the difference between a temporary market panic (such as the Oil Price Shock in 1973 and Stock Market Crash in 1987) and a persistent depression (such as the Great Depression in 1929 and the Grand Crisis in 2008). The length of the crisis duration varies greatly from temporary panic to persistent depression.

Fourth, the peaks of moments indicate that the “extreme events” do not occur randomly as predicted by stable distribution, but cluster densely in special periods (crisis).

Fifth, the length of a turbulent market depends on our observation time window. For example, the diverging period of the Great Depression was 4th/1929 to 4th/1933 with two dramatic peaks and a short dip in between which was observed from a quarterly time window in Figure 4, but the length of a turbulent market that produces the first peak was only 7 successive trading days in Table 1 for daily close data.

Current tools for diagnosing a financial crisis are mainly based on the level of deviations from the trend or percentage changes (Reinhart and Rogoff, 2009). The
higher moments of financial indicators provide a new tool in the quantitative diagnosis of a financial crisis.

A similar pattern of high moment divergence before and during the crisis can also be observed from the interest spreads (Figure 5).

Figure 5. The 3rd moments of the DJI and TED during the 2008 financial crisis. The 2nd, 4th, and 5th moments have similar patterns.

Note: TED is the interest rate spread between a three-month Eurodollar LIBOR rate and a 3 month U.S. Treasury Bill rate. Each quarterly moment is computed with 61 daily close index data within 3 months.

From Figure 5, three observations are interesting for studying financial market dynamics:

First, the interest rate movements in the international money market are closely correlated with the US stock market.

Second, the TED peak is slightly ahead of DJI peak. A dramatic rise of interest rate spread in the international money market signals a coming crisis, which is a familiar experience for money traders. For the TED spread, all the moments diverged simultaneously in 12-Sep-2008. For the logarithm DJI, only 5th moment diverged in 26-Sep-2008, where other moments grew slowly. It takes two trading days for the
TED spread to reach its peaks of moments, and more than two months for logarithm DJI. (For DJI, the largest magnitude of variance appeared in 18-Nov-2008, of other moments appeared in 15-Sep-2008. For the TED spread, all largest magnitudes of moments appeared in 28-Nov-2008.)

Third, the TED spread is more sensitive than the DJI index in market sentimental. Clearly, arbitrage activity is a double-edged sword, which could generate both negative and positive feedback in market exchanges. This is possible when market dynamics are nonlinear, since arbitrage-free opportunities only exist under linear pricing (Ross, 1976). This is the important lesson we learn from high moment analysis.

How to explain the time-evolving pattern of high moments in the financial market in theoretical perspective is the task in the next section.

II. Theoretical Models:

The Master Equation and the Birth-Death Process

A The Master Equation

The master equation is widely used in physics, chemistry, biology and finance (Kendall and Stuart, 1969; Reichl, 1998; Faller and Petruccione, 2003). The change of the time-varying probability distribution is generated by the transitions from state $x'$ to state $x$, minus the transitions from state $x$ to state $x'$. The resulted master equation is given by the following partial differential equation:

$$
\frac{\partial}{\partial t} P(x,t) = \int dx' [W(x | x',t)P(x',t) - W(x' | x,t)P(x,t)]
$$

(3)
\( P(x,t) \) is the probability distribution function, \( W(x| x', t) \) is its transition probability, represents the probability for the stochastic variable changing from state \( x' \) to \( x \) in time interval of \( t \) to \( t + dt \). For the stationary case, \( \frac{\partial}{\partial t} P(x,t) = 0 \).

A stochastic process can be defined by equation (3) by introducing a specific transition probability.

**B. The Birth and Death Process**

The name of the birth-death process was given in molecule dynamics in statistical mechanics. It can be used to study the up-down process of price movements driven by buying and selling among a large population of identical agents.

The master equation of the birth-death process in the discrete price space is the following:

\[
\frac{\partial P(x,t)}{\partial t} = W_+(x-1)P(x-1,t) + W_-(x+1)P(x+1,t) - [W_+(x) + W_-(x)]P(x,t),
\]

Where \( W_+(x) = W(x+1|x) \), \( W_-(x) = W(x-1|x) \),

The birth-death process (4) is simpler than the master equation (3) because its state space is in discrete form. The unit of price is the minimal accounted change of the variable, which is 0.01 point as the unit for a stock index, and \( \frac{0.01}{Y} \) for log-index \( S = \ln Y \).

The transition probability can be described by a vector projected in a non-orthogonal set. Its basis function is \( \{f_n : n \in \mathbb{N}\} \) with \( f_0 = 1 \), \( f_1 = x \), and \( f_n = \prod_{i=0}^{n-1}(x-i) \) for \( n \geq 2 \). Hence the transition probability \( W \) can be expressed by a vector \( (a_0, a_1, \ldots, a_n) \) in space \( (f_0, f_1, \ldots, f_n) \).
The birth-death process is the simplest model of population dynamics with identical agents in discrete state space. Its transition probability can be derived from empirical data. We will discuss it in Section III.

**C. The Linear BD Process**

The linear birth-death process is the simplest case of equation (4) when the coefficients of transition probability are: $a_i \neq 0$ and $a_i |_{i=1} = 0$.

For the linear birth-death process:

$$W_+ = bx, \quad W_- = dx$$

Here, $b$ is the birth (price-up) rate, and $d$ is the death (price-down) rate for the linear birth-death process.

Price movements can be visualized by the up and down dynamics driven by positive feedback ($W_+ = bx$) and negative feedback ($W_- = dx$) mechanisms. The trend emerges as an aggregate result of mass trading. The linear birth-death process produces a linear deterministic trend when $b - d > 0$.

$$dE(x(t)) = (b - d)E(x(t))$$

We calculate the RD for two special cases: (a) when the time limit tends to zero. (b) when its time limit tends to infinite. Their solutions are:

(7a) \[ \lim_{t \to 0} \Omega_{BD}^{t} = \sqrt{(b + d)t} \]

(7b) \[ \lim_{t \to \infty} \Omega_{BD}^{t} = \sqrt{\frac{b + d}{b - d}} \]

From equation (7a) and (7b), we can see that the short-term perspective of the linear birth-death process is a diffusion process with an explosive relative deviation,
while its long-term perspective is convergent to a steady state with a constant relative deviation. This result provides a good argument for using the birth-death process, which is capable of explaining the observed stable pattern of relative deviations in stock and macro indexes observed in Figure 3 (Chen 2002, 2005). The proofs of (6) to (7) are given in Appendix 1.

D. The Nonlinear Birth-Death Process with High Moment Expansion

In order to study market instability and financial crisis, we study a nonlinear birth-death process with 4th power of state $x$ for mathematical simplicity.

We consider a nonlinear birth-death process; its transition probability has the following form:

$$
W_+ = b_0 + b_1 f_1 + b_2 f_2 + b_3 f_3 + b_4 f_4
$$

$$
W_- = d_0 + d_1 f_1 + d_2 f_2 + d_3 f_3 + d_4 f_4
$$

Where $f_0 = 1$, $f_1 = x$, $f_2 = x(x-1)$, $f_3 = x(x-1)(x-2)$, and $f_4 = x(x-1)(x-2)(x-3)$.

Theoretically speaking, this formulation is a phenomenological description of nonlinearity up to the third power of $x$. Intuitively, the $f$ function can be visualized as consecutive trade up to four steps in a time interval by identical traders. We will see that the small nonlinearity of fourth power to $x$ is capable of understanding the mathematical condition of a market break-down or crisis.

It is hard to find the analytic solution with non-linear transition probability. But we can explore an approximation solution by means of high moment expansion. The 3rd to 5th moments are not small compared to the 2nd moment even under a linear approximation. We first try a semi-linear expansion. If the noise term is small, we may apply the method of moment expansion in Appendix 2, and obtain the Fokker-Plank equation with Poisson Representation (Gardiner 1985) for the high moments:
\[
\frac{\partial F}{\partial t} = -\frac{\partial}{\partial \alpha} [(b_1 - d_1)\alpha + (b_2 - d_2)\alpha^2 + (b_3 - d_3)\alpha^3 + (b_4 - d_4)\alpha^4]F(\alpha, t)
\]
\[
+ \frac{\partial^3}{\partial \alpha^3} [b_2\alpha + (2b_2 - d_2)\alpha^2 + (3b_3 - 2d_3)\alpha^3 + (4b_4 - 3d_4)\alpha^4]F(\alpha, t)
\]
\[
- \frac{\partial^3}{\partial \alpha^3} [b_2\alpha^2 + (3b_3 - d_3)\alpha^3 + (4b_4 - 3d_4)\alpha^4]F(\alpha, t)
\]
\[
+ \frac{\partial^4}{\partial \alpha^4} [b_2\alpha^3 + (4b_4 - d_4)\alpha^4]F(\alpha, t)
\]
\[
- \frac{\partial^5}{\partial \alpha^5} b_2\alpha^5 F(\alpha, t)
\]

Where \( F \) is no longer a real measure for high order moments. It implies that the stationary distribution may not exist.

The third and higher order moments can be solved when the noise term is small. For a stable solution, the linear case is good enough. We will show the dynamics of a turbulent market when high moments diverge. The theoretical moments would diverge at the point of \( x \) when:

\[
\frac{\partial}{\partial \alpha} [(b_1 - d_1)x + (b_2 - d_2)x^2 + (b_3 - d_3)x^3 + (b_4 - d_4)x^4] = 0,
\]

**III. The Transition Probability Observed from Empirical Data**

**A. Estimating Transition Probability in Two Periods**

In theory, the transition probability is always changing over time. This imposes a tremendous difficulty in empirical analysis of the transition probability. In econometric analysis, we can divide the available time series into several periods, and assume that the structural form is not changing over time within each period. Similarly, we may assume that the transition probability in each period is only a function of price state, not the function of time. We choose 1980 as the dividing line, since President Ronald Reagan initiated the liberalization era in market deregulation.
We wish to observe the structural change between the periods of 1950-1980 and 1981-2010.

In empirical analysis, the data frequency restricts the resolution of trading behavior. Given the frequency of data set \( \{x_t | t \in [0, +\infty)\} \) (length of \( \Delta t \) between two successive point \( x_{t+1} \) and \( x_t \)), only the net aggregate result of all trades during \( \Delta t \) accounts for the transition probability. For example, if we have the daily data, the balanced buyer-initiated and seller-initiated trades within any interval during the day have no influence on the moments calculated on a daily basis.

Assume that the transition probability doesn’t vary within the specific period. If there are \( N \) samples at a given value \( x^0 \), among which \( n_+ \) samples move up by the average jump magnitude \( \Delta x_+ \) and \( n_- \) samples move down by the average jump magnitude \( \Delta x_- \) in the next day, the transition probabilities at \( x^0 \) are

\[
W(x^0 + 1|x^0) = \frac{n_+}{N} \Delta x_+ \quad \text{and} \quad W(x^0 - 1|x^0) = \frac{n_-}{N} \Delta x_-
\]

Therefore, \( \frac{n_+}{N} \) and \( \frac{n_-}{N} \) is the probability of moving up and down, \( \Delta x_+ \) and \( \Delta x_- \) are the respective numbers of “standard” trades. We calculated the transition probability from S&P 500 daily index. Their pattern for the period of 1981-1996 and for the period of 1997-2010 is shown in Figure 6 and Figure 7 respectively.
Figure 6. The transition probabilities ($W_+$ and $-W_-$) of S&P 500 daily close from 1950-1980. The main curves of $W_+$ and $-W_-$ (except the segments between 110 to 140 points) are not far from linear.

Figure 7. In 1981-2010, the transition probabilities ($W_+$ and $-W_-$) of S&P 500 index are marked by two curves, which are highly non-linear with two visible humps or dips.
Comparing Figure 6 with Figure 7 we have three observations:

First, the transition probability has two curves: both curves are not straight lines. It is a clear feature of nonlinear dynamics in the birth-death process. The upper curve indicates a price-up magnitude driven by positive feedback and the lower curve indicates a price-down magnitude driven by negative feedback in market trading. The observed behavior is more complex than the noise trader model (De Long, Shleifer, Summers, and Waldmann, 1990).

Second, the upper and lower curves are not symmetric, since there is a growth trend in the market index time series.

Third, there is remarkable difference between the two periods. For period I of 1980-1996 without severe crisis, the two transition probability curves are more or less balanced with only gradual changes. For period II of 1997-2010 with the 2008 crisis, the transition probability curves have visible humps in the upper curve and dips in the lower curve. We can see the behavior changes that occurred within some price range. The polarized pattern appears in public opinion on future market trends (Chen, 1991). The psychological source of the market instability was characterized as “animal spirits” (Keynes, 1936; Akerlof and Shiller, 2009).

B. Estimating Transition Probability Coefficients

The following simplified equations are used to calculate the transition probability from \( Y(t) \) to estimate the coefficients of \( W_+ \) and \( W_- \)

\[
W_+ = b_0 + b_1 Y + b_2 Y^2 + b_3 Y^3 + b_4 Y^4 \\
W_- = d_0 + d_1 Y + d_2 Y^2 + d_3 Y^3 + d_4 Y^4
\]  

(11)

The coefficients of \( W_+ \) and \( W_- \) from the S&P500 index during 1997-2010 is shown in Table 2.
Table 2. The coefficient of the transition probability of S&P500 Index (original daily close) for the period of 1997-2010

<table>
<thead>
<tr>
<th>Time</th>
<th>The polynomial fit coefficient</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997-2010</td>
<td>$b_0 = -101.4$</td>
<td>$b_1 = 0.5011$</td>
</tr>
<tr>
<td></td>
<td>$d_0 = -625.9$</td>
<td>$d_1 = 2.277$</td>
</tr>
</tbody>
</table>

In table 2, the period 1997-2010 is chosen for mathematical simplicity of curve fitting. We will show the transition probability faithfully recorded in the recent subprime crisis in the next section.

IV. Estimating the Condition of Market Break-Down

We refer to the market as turbulent when equation (10) is valid and the probability distribution no longer exists. This situation is similar to the critical fluctuation in statistical physics (Gardiner, 1985). It implies:

\[
\sigma, \xi, \nu, \xi^5, \text{ and higher moments} \rightarrow \infty
\]

Where $\sigma^2$ is variance, $\xi^3$ is the 3rd moment, $\nu^4$ the 4th moment, and $\xi^5$ is the 5th moment.

Under this situation, we have four implications: First, all statistic variables become meaningless. Second, the market trend collapsed. In other words, market expectations have no consensus, only panic rules the market. Third, the condition (10) reveals the possibility of market breakdowns. Fourth, the crisis region is beyond the scope of the nonlinear birth-death process. We need a more advanced tool to describe the dynamic process in a turbulent market. This knowledge is absent in linear models of finance theory.

The nonlinear pattern of transition probability reveals the dynamic trend of population dynamics. Since we know the coefficients of the transition probability, we could estimate the location of the market crisis. If the positive coefficients represent
the degree of positive feedback and the negative coefficients the degree of negative feedback, the difference between positive and negative feedback measures the net movement within a day, we refer to it as the net daily change rate.

Keynes and behavioral economists pointed out the role of mass psychology (Keynes, 1936; Thaler, 2005). We can visualize the market tide driven by collective psychology as a curve. Its falling segment represents a market tide towards equilibrium while a rising segment signals a market tide towards disequilibrium. Figure 8 shows the numerical results of the net daily change rate, which indicates a down – up – down market tides. The up phase describes a collective fad for a market bubble. We assume that the turning point from the up to the down phase may generate a market breakdown or crisis.

Figure 8. The curve of a changing market tide in terms of the net daily change rate (1997-2010).

*Note:* The curve is calculated from the 4th degree polynomial fitted transition probability from 1997-2010. The up segment indicates two hot-speculation periods from 950 to 1229 in 1997-2000 and 2003-2008, and panic in 2008. The vertical line marks the turning point in market psychology, which is 1229 in the price level. Therefore, we can identify the date from the historical time series.
From Figure 8, the turning point can be estimated from the numerical solution of the nonlinear birth-death process. According to historical data, the S&P 500 index closed at 1209.13 on 25-Sep-2008, the day before high moments of DJI diverged. Both the estimated crisis date based on Figure 8 and the calculated high moment explosion date in Table 1 are remarkably close to the real event.

Certainly, the transition probability can only be estimated ex post. In practice, the rising tide signals an upcoming bubble in the financial market. This is a valuable indicator for the early warning of a crisis.

For empirical observation, a turbulent market can be observed from high frequency data with a sudden rise of trading volume or even a market frozen in a period of hours or days. But a turbulent market may not exist long when government intervention calms the market in a modern economy. Our approach is capable of identifying the period of market turbulence, which creates the space for government interference.

V. Conclusion

There is a widely perceived image that neoclassical economics is imitating equilibrium physics (Mirowski, 1989). This is partially true in a philosophical paradigm because both of them are belong to the equilibrium school of theoretical thinking, but not true in the mathematical formulation for empirical analysis. For example, the Brownian motion model in physics is molecule dynamics with many particles, but option pricing based on geometric Brownian motion is a representative agent model with only one particle (Einstein, 1926; Black and Scholes, 1973). Statistical mechanics starts with infinite moments with time-varying distribution probability, but econometric analysis is based on the i.i.d. model with a finite mean and variance. Most scientists agree that economic phenomena are more complex than physics and chemistry. Strangely, economic models are much simpler than models in physics and chemistry. Paul Krugman, the 2008 Nobel Laureate in economics, recently wrote a provocative article in the New York Times, titled “How Economists
“Got It So Wrong?” His answer was “mistaken beauty for truth” (Krugman, 2009). However, there are no scientific criteria to assert that linear models are prettier than nonlinear ones (Galbraith, 2009). The fundamental issue here is not in the mathematical beauty but the empirical relevance of economic theory. We know that existing linear models are not capable of explaining market turbulence and financial crisis. To paraphrase Krugman, we demonstrate that equilibrium models in finance are “mistaken simplicity for complexity” by ignoring nonlinear dynamics, high moment deviations, time-varying distribution, and social interactions. Our progress was made by integrating these complex factors into properly formulated economic dynamics. Our inspiration came from Einstein, when he discovered that non-Euclidean geometry was more relevant for general relativity than Euclidean geometry.

A general framework with more advanced mathematical representations is needed to solve old controversies in economics. In this article, we show that the time-varying probability distribution and the birth-death process are useful in diagnosing the nature of business cycles and financial crisis.

The empirical observations through high moments clearly indicate two market regimes appear alternatively: there is a calm market when the high moments are much smaller than the second moment (variance); or there is a turbulent market when its high moments are comparable or even larger than the normal second moment.

From empirical transition probability, we found that both negative and positive trading behavior coexists in calm and turbulent periods. This picture is different from equilibrium economics and behavioral finance, since the former only considers the stabilizing role of negative feedback and the latter emphasizes the destabilizing role of positive feedback. The empirical patterns of transition probability reveal two dynamic mechanisms in price dynamics: the quasi equilibrium process when positive and negative transition curves are smooth and near symmetric; the disequilibrium process when its positive and negative transition curves are S-shaped and significantly asymmetric. By two-period analysis of empirical data, we found a visible link between liberalization policy and financial crisis. This result is very
different from the exogenous school with fat tails (Laherrère and Sornette, 1999; Barror and Ursúa, 2008; Gabaix, Gopikrishnan, Plerou and Stanley, 2006).

The birth-death process is simple enough to explain the origin of the viable market with persistent fluctuations even during a crisis period: the stable pattern of the long-term relative deviation results from the collective action in a population model. The nonlinear birth-death process is a useful model, which integrates the calm market regime for the equilibrium school and the turbulent market regime for the disequilibrium school. Its nonlinear pattern of transition probability demonstrates the nonlinear nature of endogenous instability in business cycles. Its theoretical relevance lies in the fact that it is capable of estimating the condition of a market breakdown. Its implied date is close to the historical event of the U.S. 2008 financial crisis.

Philosophically speaking, both schools of thoughts reflect some aspects of complex dynamics of the real market. The so-called efficient market in finance literature is essentially a simplified model of the calm market, which has three observable features: First, the growth trend from stock market indexes can be ignored in the short-term perspective in financial econometrics. Second, the higher moments are quite small in comparison with the variance. Therefore, the price movements can be considered as a diffusion process with a Gaussian distribution or i.i.d. Third, the diffusion process is a special case of the linear birth-death process with a short-term perspective, whose relative deviation is explosive. It implies that the diffusion process, i.e. in the form of pattern of the stable relative deviation (Chen, 2002, 2005, 2010; Li 2002). On the other hand, the noise trader the random walk model or the geometric Brownian motion, is not capable of describing a “viable market” with an observed model in behavioral finance is also a simplified model with two discrete periods when a positive feedback and negative feedback trading strategy dominate the market alternatively (De Long, Shleifer, Summers, and Waldmann, 1990). These two trading strategies can be considered as special cases in our nonlinear birth-death process.

The policy implications from our analysis are self-evident. Market bubbles and financial crisis occurred in the period of 1981-2010, which reveals the possible link between the liberalization policy started by President Reagan and the 2008 financial
crisis. Clearly, price level alone is not enough to gauge market sentiment. Dramatically rising high moments signal the emergence of speculative bubbles or animal spirits. Regulating market leverage in trading may be useful in preventing possible crisis.

Methodologically speaking, the population dynamics of the birth-death process provides a useful framework for studying complex financial dynamics including calm and turbulent market as well as market resilience after crisis. Comparing to existing approaches in parametric econometrics and computer simulation based on a heterogeneous agent, the high moment and transition probability provides an effective approach both in the empirical analysis and theoretical understanding of market instability and financial crisis. Our population model of identical agents provides a simpler explanation of market bubbles than the heterogeneous agent model. Nonlinearity and population behavior play key roles in the genesis of financial crisis, which is beyond the scope of linear dynamics and representative agent models. We are developing a general model of option pricing based on the master equation and birth-death process. We will address this issue elsewhere.

Appendix 1. The Relative Deviation of the Linear Birth-Death Process

A linear birth-death process can be simply approximated by a deterministic process $E(x(t))$ plus a stochastic process $z$ (van Kampen, 1976). The deterministic trend is

$$\frac{dE(x)}{dt} = W_+ [E(x)] - W_- [E(x)] ,$$

and the Fokker-Plank equation for variance is

$$\frac{\partial P(z,t)}{\partial t} = -(W_+ [E(x)] - W_- [E(x)]) \frac{\partial}{\partial z} zP(z,t)$$

$$+ \frac{1}{2} \frac{\partial^2}{\partial z^2} (W_+ [E(x)] + W_- [E(x)])P(z,t).$$
Where
\[ \{z(t), z(t)\} = \int_0^t dt'(W_1[E(x(t'))] + W_2[E(x(t'))]) \exp \left\{ 2 \int_0^t (W_1'[E(x(s))] - W_2'[E(x(s))]) ds \right\} \]
is the variance of \( x_t \). Hence the volatility \( \sigma_t^2 \) has a value evolves with the expectation \( E(x(t)) \)
\begin{equation}
\sigma_t^2 = \frac{b + d}{b - d} E(x(t)) \left[ e^{(b - d)t} - 1 \right]
\end{equation}
Then the RD of linear BD process is \( \Omega_{BD} = \frac{b + d}{b - d} \sqrt{1 - e^{(b - d)t}} \), which has three special cases:

\begin{align*}
\lim_{t \to \infty} \Omega_{BD} & = \frac{b + d}{b - d}, \\
\lim_{t \to 0} \Omega_{BD} & = \sqrt{(b + d)t}, \\
\lim_{b \to d} \Omega_{BD} & = \sqrt{(b + d)t}.
\end{align*}

Note that here \( \Omega_{BD} \) is the RD for \( Y(t) \). For logarithmic series \( S \), \( \lim_{t \to \infty} \Omega_{BD}(S) \) and \( \lim_{b \to d} \Omega_{BD}(S) \) are also growing with \( \sqrt{t} \). And if \( \Omega_{BD}(Y)\big|_{t \to 0} \) is stable, empirical \( \Omega_{BD}(S)\big|_{t \to 0} \) is stable with only a slight decrease, as shown in figure 3 (the proof of the similarity of \( \Omega_{BD}(Y) \) and \( \Omega_{BD}(S) \) will be published elsewhere). Therefore, the linear Birth-death process can describe the stable RD in both original and logarithmic economic indexes.

**Appendix 2. The Generating Function and Moment Expansion**

We introduce a generating function \( G \)
\( G(s,t) = \sum_x s^x P(x,t) \)

(17)

\[ \partial_s G = \partial_s^* G + \partial_t^* G, \]

Where

\[ \partial_s^* G = \sum_x s^x [W_x(x-1)P(x-1,t) - W_x(x)P(x,t)] \]

(18)

\[ \partial_t^* G = \sum_x s^x [W_x(x+1)P(x-1,t) - W_x(x)P(x,t)] \]

Take note of that \( \frac{x^1}{(x-N)!} s^x = s^N \partial_s^N s^x \) and \( P(x-1,t) = P(x,t) \), we get

\[ \partial_s^* G = \sum_x [b_1(s^2-s)\partial_s^2 + b_2(s^3-s^2)\partial_s^3 + b_3(s^4-s^3)\partial_s^4]s^x P(x,t) \]

(19)

\[ = [b_1(s^2-s)\partial_s^2 + b_2(s^3-s^2)\partial_s^3 + b_3(s^4-s^3)\partial_s^4 + b_4(s^5-s^4)\partial_s^5] s^1 P(x,t) \]

And

\[ \partial_t^* G = -\sum_x [d_1(s-1)\partial_s + d_2(s^2-s)\partial_s^2 + d_3(s^3-s^2)\partial_s^3 + d_4(s^4-s^3)\partial_s^4] s^1 P(x,t) \]

(20)

\[ = [-d_1(s-1)\partial_s + d_2(s^2-s)\partial_s^2 + d_3(s^3-s^2)\partial_s^3 + d_4(s^4-s^3)\partial_s^4] s^1 P(x,t) \]

We use a Poisson expression to expand equation (17). Suppose the distribution function \( P(x,t) \) can be expanded with a Poisson series

\[ P(x,t) = \int d\alpha \frac{e^{-\alpha} \alpha^x}{x!} F(\alpha, t). \]

(21)

Note that \( F(\alpha, t) \) is by no means a measure, it is used to create the moments as a substitute description of the distribution \( P(x,t) \). The relationship \( E(x) = E(\alpha) \) can be derived from the direct calculation of \( E(x^i) \).
\[ E(x^j) = \sum_x \frac{x!}{(x-j)!} \frac{e^{-\alpha}}{x!} F(\alpha, t) d\alpha \]
\[ = \int \alpha^j e^{-\alpha} e^{\alpha} F(\alpha, t) d\alpha \]
\[ = \int \alpha^j F(\alpha, t) d\alpha = E(\alpha^j) \]

The generating function becomes

\[ G(s, t) = \int e^{(s-1)\alpha} F(\alpha, t) d\alpha \]

Notice \( \alpha G = \frac{\partial G}{\partial s} \) and \( sG = (-\frac{\partial}{\partial \alpha} + 1)G \), equation (22) becomes

\[ \frac{\partial}{\partial \alpha} G = \int d\alpha \exp[(s-1)\alpha] F(\alpha, t) \left[ \frac{\partial}{\partial \alpha} (b_1(\frac{\partial}{\partial \alpha} + 1) - d_1)\alpha + \frac{\partial}{\partial \alpha} (b_2(\frac{\partial}{\partial \alpha} + 1) - d_2)\alpha^2 \right. \]
\[ \left. + \frac{\partial}{\partial \alpha} (\frac{\partial}{\partial \alpha} + 1)^2 (b_3(\frac{\partial}{\partial \alpha} + 1) - d_3)\alpha^3 + \frac{\partial}{\partial \alpha} (\frac{\partial}{\partial \alpha} + 1)^3 (b_4(\frac{\partial}{\partial \alpha} + 1) - d_4)\alpha^4 \right] . \]

Integrate equation (24) by parts; we have the Fokker-Plank equation with Poisson Representation

\[ \frac{\partial F}{\partial t} = -\frac{\partial}{\partial \alpha} [(b_1 - d_1)\alpha + (b_2 - d_2)\alpha^2 + (b_3 - d_3)\alpha^3 + (b_4 - d_4)\alpha^4] F(\alpha, t) \]
\[ + \frac{\partial^2}{\partial \alpha^2} [b_1\alpha^2 + (2b_2 - d_2)\alpha^3 + (3b_3 - 2d_3)\alpha^4 + (4b_4 - 3d_4)\alpha^5] F(\alpha, t) \]
\[ - \frac{\partial^3}{\partial \alpha^3} [b_2\alpha^3 + (3b_3 - d_3)\alpha^4 + (6b_4 - 3d_4)\alpha^5] F(\alpha, t) \]
\[ + \frac{\partial^4}{\partial \alpha^4} [b_3\alpha^4 + (4b_4 - d_4)\alpha^5] F(\alpha, t) \]
\[ - \frac{\partial^5}{\partial \alpha^5} b_4\alpha^5 F(\alpha, t) \]

With

\[ \frac{dE(x)}{dt} = \frac{d\alpha}{dt} = (b_1 - d_1)x + (b_2 - d_2)x^2 + (b_3 - d_3)x^3 + (b_4 - d_4)x^4 \]
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35


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Summary

High moments and transition probability provide new tools for diagnosing a financial crisis. Both calm and turbulent markets can be explained by the birth-death process for up-down price movements driven by identical agents.

In addition to the optimization approach of a representative agent and the simulation approach of heterogeneous agents, we develop a third approach of social dynamics, i.e. statistical mechanics in socio-economic systems with identical agents. We introduce the birth-death process as the benchmark model for up-down price movements, which is capable of explaining market resilience and a financial crisis. We develop numerical algorithms for calculating high moment deviations and estimate the transition probability from an empirical stock index series. High (3rd to 5th) moments rapidly rise before and during the crisis. Changing patterns in trading behavior can be observed from S-shaped curves in the transition probability. The theoretical condition of a market breakdown can be obtained from the solution of the master equation with a high moment expansion. The equilibrium model of the efficient market can be considered as a special case of a calm market when high moment deviations are very small compared to the variance. The noise trader model in behavioral finance can be justified by the rising curve of the transition probability with identical agents. We find solid empirical evidence for the endogenous nature of business cycles and market instability. Diversification strategy failed during the sub-prime crisis because the mean-variance approach ignores high moment deviations. We demonstrate the link between the liberalization policy and the recent crisis. These results are beyond the scope of neoclassical models in finance theory, which is based on the representative agent model of random walk or geometric Brownian motion.

Our method is an empirical analysis based on the time-varying distribution probability. High moments and the transition probability are derived from a
moving time window. We do not make ad hoc assumptions about the transition probability that is common in equilibrium models.

We made three general assumptions about our theoretical model. First, we have a population of identical agents. Second, trading is conducted in a discrete price space and continuous time. Third, the transition probability is only the function of price but independent of time during a specific time period, which is similar to a multi-stage econometric analysis. We found that a calm market characterized the period of 1950-1980 and a turbulent market appeared during the period of 1981-2010.

The relevance of economic analysis is based on two criteria: First, empirical evidence for theoretical implications; Second, a general framework for seemingly conflicting features. We demonstrate that the birth-death process is better than the representative agent models like random walk and Brownian motion in these two aspects. The crisis condition for a market breakdown can be derived from the solution of the nonlinear birth-death process. Our numerical estimation of a market turning point is close to the historical event of the U.S. 2008 financial crisis. We have a new understanding of the seemingly contradictory phenomena including the market resilience and a recurrent crisis. The valuable insights from equilibrium and disequilibrium schools can be integrated into a more general theory of economic complexity with many interacting agents. Statistical mechanics provides a powerful tool in analyzing economic dynamics when its transition probability is derived from empirical data without ad hoc assumption of a Gaussian type distribution.

There is a widely perceived image that neoclassical economics is imitating equilibrium physics in conservative systems. This is partially true in equilibrium thinking, but not true in mathematical formulation. For example, the Einstein model of the Brownian motion in physics is a molecule dynamics with many particles, but option pricing based on geometric Brownian motion in finance theory is a representative agent model with only one particle. Statistical mechanics starts with infinite moments with a time-varying distribution
probability, but econometric analysis is based on the i.i.d. model with a finite mean and variance. Most scientists agree that economic phenomena are more complex than physics and chemistry. Strangely, economic models are much simpler than simple models in physics and chemistry. The mathematical formulation of a self-stabilizing market was created by ignoring nonlinear dynamics, high moment deviations, time-varying distribution, and social interactions. Our progress was made by integrating these complex factors into properly formulated economic dynamics. Our inspiration came from Einstein, when he discovered that the non-Euclidean geometry was more relevant for general relativity than Euclidean geometry.

The equilibrium models of the efficient market and option pricing are based on the representative agent model of random walk and geometric Brownian motion. They do not provide any clue on the condition of financial crisis. Our general framework greatly extends the scope of equilibrium models in finance. We have a new understanding of the seemingly contradictory phenomena including market resilience and recurrent crisis. The valuable insights from the equilibrium and disequilibrium schools can be integrated into a more general theory of economic complexity with many interacting agents. Statistical mechanics provides a powerful tool in analyzing economic dynamics when its transition probability is derived from empirical data to address new features in open social dynamics.

Our conclusions are: a financial crisis can be understood by nonlinear interactions among identical agents. The endogenous nature and the nonlinear mechanism can be diagnosed from an empirical analysis of high moments and the transition probability within a moving time window. Social dynamics based on the master equation and the birth-death process provides a better tool for analyzing business cycles and a financial crisis. Our approach is more general than the representative agent model in theoretical modeling and much simpler than the simulation approach with heterogeneous agents in empirical analysis.